ASYMPTOTIC ENERGY DEPENDENCE OF HADRONIC TOTAL CROSS SECTIONS FROM LATTICE QCD

Matteo Giordano^{1,*}, Enrico Meggiolaro^{2,†}, Niccolò Moretti^{2‡}

Departamento de Física Teórica, Universidad de Zaragoza, Calle Pedro Cerbuna 12, E-50009 Zaragoza, Spain

² Dipartimento di Fisica, Università di Pisa, and INFN, Sezione di Pisa, Largo Pontecorvo 3, I–56127 Pisa, Italy

Abstract

The nonperturbative approach to soft high–energy hadron–hadron scattering, based on the analytic continuation of Wilson–loop correlation functions from Euclidean to Minkowskian theory, allows to investigate the asymptotic energy dependence of hadron–hadron total cross sections in lattice QCD. In this paper we will show, using best fits of the lattice data with proper functional forms satisfying unitarity and other physical constraints, how indications emerge in favor of a universal asymptotic high–energy behavior of the kind $B \log^2 s$ for hadronic total cross sections.

^{*}E-mail: giordano@unizar.es

[†]E-mail: enrico.meggiolaro@df.unipi.it ‡E-mail: nicco.moretti@gmail.com

1. Introduction

The problem of predicting total cross sections at high energy from first principles is one of the oldest open problems of hadronic physics, not yet satisfactorily solved in QCD. Present-day experimental observations (up to a center-of-mass total energy \sqrt{s} 7 TeV, reached at the LHC pp collider [1]) seem to support the following asymptotic high-energy behavior: $\sigma_{\text{tot}}^{(hh)}(s) \sim B \log^2 s$, with a universal (i.e., not depending on the particular hadrons involved) coefficient $B \simeq 0.3$ mb [2]. This behavior is consistent with the well-known Froissart-Lukaszuk-Martin (FLM) theorem [3], according to which, for $s \to \infty$, $\sigma_{\text{tot}}^{(hh)}(s) \le \frac{\pi}{m_{\pi}^2} \log^2\left(\frac{s}{s_0}\right)$, where m_{π} is the pion mass and s_0 is an unspecified squared mass scale. (Let us observe that the experimental value of B is much smaller than [about 0.5%] the coefficient $\frac{\pi}{m_{\pi}^2}$ appearing in the FLM bound.) As we believe QCD to be the fundamental theory of strong interactions, we also expect that it correctly predicts from first principles the behavior of hadronic total cross sections with energy. Anyway, in spite of all the efforts, a satisfactory solution to this problem is still lacking. Theoretical supports to the universality of the coefficient B were found in the model of the iteration of soft-Pomeron exchanges by eikonal unitarization [4] (recently revisited in the context of holographic QCD [5]), and also using arguments based on the so-called Color Glass Condensate of QCD [6], or simply modifying the original Heisenberg's model [7] in connection with the presence of glueballs [8].

This problem is part of the more general problem of high–energy elastic scattering at low transferred momentum, the so–called soft high–energy scattering. As soft high–energy processes possess two different energy scales, the total center–of–mass energy squared s and the transferred momentum squared t, smaller than the typical energy scale of strong interactions ($|t| \lesssim 1~{\rm GeV}^2 \ll s$), we cannot fully rely on perturbation theory. A genuine nonperturbative approach in the framework of QCD has been proposed in [9] and further developed in a number of papers (see, e.g., [10] for a review and a complete list of references): using a functional integral approach, high–energy hadron–hadron elastic scattering amplitudes are shown to be governed by the correlation function (CF) of certain Wilson loops defined in Minkowski space [11, 12, 13, 14, 15]. Moreover, as it has been shown in [16, 17, 18], such a CF can be reconstructed by analytic continuation from the CF of two Euclidean Wilson loops, that can be calculated using the nonperturbative methods of Euclidean Field Theory.

The analytic-continuation relations have allowed the nonperturbative investigation of

correlators (and the corresponding scattering amplitudes) using some analytical models, such as the Stochastic Vacuum Model (SVM) [19], the Instanton Liquid Model (ILM) [20, 21], the AdS/CFT correspondence [22] and, finally, they have also allowed a numerical study by Monte Carlo simulations in Lattice Gauge Theory (LGT) [23, 21] (see also Refs. [24] for a short review). Although the numerical results obtained on the lattice can be considered "exact" (since they are derived from first principles of QCD), it is not possible to relate them directly to physical quantities, since the analytic continuation of the correlator can be performed only if an analytical dependence on the variables is known, while lattice data can be obtained only for a discrete (finite) set of values. However, it is possible to test the goodness of the known analytical models (like the ones we have mentioned above) simply through a best fit to the lattice data. This analysis has already been done in Refs. [23, 21] and it is briefly recalled in Section 3. The result of this analysis is not, generally speaking, satisfactory: known analytical models lead to bad quality best fits and, moreover, none of them provides a physically acceptable total cross section.

In this paper, after a brief survey (for the benefit of the reader) of the nonperturbative approach to soft high-energy scattering in the case of meson-meson elastic scattering (in Section 2), and of the numerical approach based on LGT, comparing the numerical results to the existing analytical models (in Section 3), we will concentrate on the search for a new parameterization of the (Euclidean) correlator that, in order: i) fits well the lattice data; ii) satisfies (after analytic continuation) the unitarity condition; and, most importantly, iii) leads to a rising behavior of total cross sections at high energy, in agreement with experimental data. In particular, one is interested in the dependence of the CF on the angle θ between the loops, since it is related, after analytic continuation, to the energy dependence of the scattering amplitudes, and also in its dependence on the impact parameter distance. In Section 4 we show that, making some reasonable assumptions about the angular dependence and the impact-parameter dependence of the various terms in the parameterization, our approach leads quite "naturally" to total cross sections rising asymptotically as $B \log^2 s$ (that is what experimental data seem to suggest). Moreover, in our approach the coefficient B turns out to be universal, i.e, the same for all hadronic scattering processes (as it also seems to be suggested by experimental data), being related to the mass-scale μ which sets the large impact-parameter behavior of the correlator. This is actually the main result of this paper. In Section 5 we draw our conclusions and discuss some prospects for the future.

2. High-energy meson-meson elastic scattering amplitude and Wilson-loop correlation functions

We sketch here the nonperturbative approach to soft high–energy scattering (see [23] for a more detailed presentation). The elastic scattering amplitude $\mathcal{M}_{(hh)}$ of two hadrons, or more precisely mesons (taken for simplicity with the same mass m), in the soft high–energy regime can be reconstructed, after folding with two proper squared hadron wave functions $|\psi_1|^2$ and $|\psi_2|^2$, describing the two interacting hadrons, from the scattering amplitude $\mathcal{M}_{(dd)}$ of two dipoles of fixed transverse sizes $\vec{R}_{1,2\perp}$, and fixed longitudinal–momentum fractions $f_{1,2}$ of the two quarks in the two dipoles [11, 12, 13, 14, 15]:

$$\mathcal{M}_{(hh)}(s,t) = \int d^2 \vec{R}_{1\perp} \int_0^1 df_1 |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 |\psi_2(\vec{R}_{2\perp}, f_2)|^2 \times \mathcal{M}_{(dd)}(s, t; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2),$$
(2.1)

with: $\int d^2 \vec{R}_{1\perp} \int_0^1 df_1 |\psi_1(\vec{R}_{1\perp}, f_1)|^2 = \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 |\psi_2(\vec{R}_{2\perp}, f_2)|^2 = 1.$

(For the treatment of baryons, a similar, but more involved, picture can be adopted, using a genuine three–body configuration or, alternatively and even more simply, a quark–diquark configuration: we refer the interested reader to the above–mentioned original references [11, 12, 13, 14, 15].)

In turn, the dipole–dipole (dd) scattering amplitude is obtained from the (properly normalized) CF of two Wilson loops in the fundamental representation, defined in Minkowski spacetime, running along the paths made up of the quark and antiquark classical straight–line trajectories, and thus forming a hyperbolic angle $\chi \simeq \log(s/m^2)$ in the longitudinal plane (see Fig. 1). The paths are cut at proper times $\pm T$ as an infrared regularization, and closed by straight–line "links" in the transverse plane, in order to ensure gauge invariance. Eventually, the limit $T \to \infty$ has to be taken. It has been shown in [16, 17, 18] that the relevant Minkowskian CF $\mathcal{G}_M(\chi; T; \vec{z}_\perp; 1, 2)$ (\vec{z}_\perp being the impact parameter, i.e., the transverse separation between the two dipoles) can be reconstructed, by means of analytic continuation, from the Euclidean CF of two Euclidean Wilson loops,

$$\mathcal{G}_{E}(\theta; T; \vec{z}_{\perp}; 1, 2) \equiv \frac{\langle \widetilde{\mathcal{W}}_{1}^{(T)} \widetilde{\mathcal{W}}_{2}^{(T)} \rangle_{E}}{\langle \widetilde{\mathcal{W}}_{1}^{(T)} \rangle_{E} \langle \widetilde{\mathcal{W}}_{2}^{(T)} \rangle_{E}} - 1, \quad \widetilde{\mathcal{W}}_{1,2}^{(T)} \equiv \frac{1}{N_{c}} \text{Tr} \left\{ T \exp \left[-ig \oint_{\widetilde{\mathcal{C}}_{1,2}} \widetilde{A}_{\mu}(\tilde{x}) d\tilde{x}_{\mu} \right] \right\}, \tag{2.2}$$

where $\langle ... \rangle_E$ is the average in the sense of the Euclidean QCD functional integral, and the arguments "1[2]" in \mathcal{G}_E (and \mathcal{G}_M) stand for " $\vec{R}_{1[2]\perp}$, $f_{1[2]}$ ". The Euclidean Wilson loops

 $\widetilde{\mathcal{W}}_{1,2}^{(T)}$ are calculated on the following quark [q] – antiquark $[\bar{q}]$ straight–line paths,

$$\widetilde{C}_1: \widetilde{X}_1^{q[\bar{q}]}(\tau) = \widetilde{z} + \frac{\widetilde{p}_1}{m}\tau + f_1^{q[\bar{q}]}\widetilde{R}_1, \quad \widetilde{C}_2: \widetilde{X}_2^{q[\bar{q}]}(\tau) = \frac{\widetilde{p}_2}{m}\tau + f_2^{q[\bar{q}]}\widetilde{R}_2, \tag{2.3}$$

with $\tau \in [-T, T]$, and closed by straight-line paths in the transverse plane at $\tau = \pm T$. The four-vectors \tilde{p}_1 and \tilde{p}_2 are chosen to be $\tilde{p}_{1,2} = m(\pm \sin \frac{\theta}{2}, \vec{0}_{\perp}, \cos \frac{\theta}{2})$, θ being the angle formed by the two trajectories, i.e., $\tilde{p}_1 \cdot \tilde{p}_2 = m^2 \cos \theta$. Moreover, $\tilde{R}_i = (0, \vec{R}_{i\perp}, 0)$, $\tilde{z} = (0, \vec{z}_{\perp}, 0)$ and $f_i^q \equiv 1 - f_i$, $f_i^{\bar{q}} \equiv -f_i$. We define also the Euclidean and Minkowskian CFs with the infrared cutoff removed as

$$\mathcal{C}_{E}(\theta; \vec{z}_{\perp}; 1, 2) \equiv \lim_{T \to \infty} \mathcal{G}_{E}(\theta; T; \vec{z}_{\perp}; 1, 2),
\mathcal{C}_{M}(\chi; \vec{z}_{\perp}; 1, 2) \equiv \lim_{T \to \infty} \mathcal{G}_{M}(\chi; T; \vec{z}_{\perp}; 1, 2).$$
(2.4)

The dd scattering amplitude is then obtained from $C_E(\theta;...)$ [with $\theta \in (0,\pi)$] by means of analytic continuation as

$$\mathcal{M}_{(dd)}(s,t;1,2) \equiv -i \ 2s \int d^2 \vec{z}_{\perp} e^{i\vec{q}_{\perp} \cdot \vec{z}_{\perp}} \mathcal{C}_M(\chi \simeq \log(s/m^2); \vec{z}_{\perp};1,2)$$

$$= -i \ 2s \int d^2 \vec{z}_{\perp} e^{i\vec{q}_{\perp} \cdot \vec{z}_{\perp}} \mathcal{C}_E(\theta \to -i\chi \simeq -i \log(s/m^2); \vec{z}_{\perp};1,2) , \qquad (2.5)$$

where s and $t = -|\vec{q}_{\perp}|^2$ (\vec{q}_{\perp} being the transferred momentum) are the usual Mandelstam variables (for a detailed discussion on the analytic continuation see [18], where we have shown, on nonperturbative grounds, that the required analyticity hypotheses are indeed satisfied). By virtue of the *optical theorem* and of Eqs. (2.1) and (2.5), the total cross section is then given by the expression

$$\sigma_{\text{tot}}^{(hh)}(s) \underset{s \to \infty}{\sim} \frac{1}{s} \text{Im} \mathcal{M}_{(hh)}(s, t = 0)$$

$$= -2 \int d^{2}\vec{R}_{1\perp} \int_{0}^{1} df_{1} |\psi_{1}(\vec{R}_{1\perp}, f_{1})|^{2} \int d^{2}\vec{R}_{2\perp} \int_{0}^{1} df_{2} |\psi_{2}(\vec{R}_{2\perp}, f_{2})|^{2}$$

$$\times \int d^{2}z_{\perp} \operatorname{Re} \mathcal{C}_{E}(\theta \to -i\chi \simeq -i\log(s/m^{2}); \vec{z}_{\perp}; \vec{R}_{1\perp}, f_{1}, \vec{R}_{2\perp}, f_{2}). \tag{2.6}$$

If one chooses hadron wave functions invariant under rotations and under the exchange $f_i \to 1 - f_i$ (see Refs. [14, 15] and also [10], §8.6, and references therein), the CF C_E in Eqs. (2.1) and (2.6) can be substituted (without changing the result) with the following

averaged CF $[\vec{R}_{i\perp} = |\vec{R}_{i\perp}|(\cos\phi_i, \sin\phi_i)]$:

$$\mathcal{C}_{E}^{ave}(\theta; |\vec{z}_{\perp}|; |\vec{R}_{1\perp}|, f_{1}, |\vec{R}_{2\perp}|, f_{2}) \equiv \int \frac{d\phi_{1}}{2\pi} \int \frac{d\phi_{2}}{2\pi} \times \frac{1}{4} \left\{ \mathcal{C}_{E}(\theta; \vec{z}_{\perp}; \vec{R}_{1\perp}, f_{1}, \vec{R}_{2\perp}, f_{2}) + \mathcal{C}_{E}(\theta; \vec{z}_{\perp}; \vec{R}_{1\perp}, 1 - f_{1}, \vec{R}_{2\perp}, f_{2}) + \mathcal{C}_{E}(\theta; \vec{z}_{\perp}; \vec{R}_{1\perp}, f_{1}, \vec{R}_{2\perp}, f_{2}) + \mathcal{C}_{E}(\theta; \vec{z}_{\perp}; \vec{R}_{1\perp}, f_{1}, \vec{R}_{2\perp}, f_{2}) + \mathcal{C}_{E}(\theta; \vec{z}_{\perp}; \vec{R}_{1\perp}, f_{1}, \vec{R}_{2\perp}, f_{2}) \right\}. \quad (2.7)$$

We note here that, as a consequence of the (Euclidean) crossing-symmetry relations [25], $C_E(\pi - \theta; \vec{z}_\perp; 1, 2) = C_E(\theta; \vec{z}_\perp; 1, \overline{2}) = C_E(\theta; \vec{z}_\perp; \overline{1}, 2)$, where the arguments " \overline{i} " stand for " $-\vec{R}_{i\perp}, 1 - f_i$ " (i = 1, 2), the function C_E^{ave} is automatically crossing-symmetric, i.e., $C_E^{ave}(\pi - \theta; \ldots) = C_E^{ave}(\theta; \ldots)$ for fixed values of the other variables. (The exchange "1, 2" \rightarrow "1, $\overline{2}$ ", or "1, 2" \rightarrow " $\overline{1}$, 2", as well as $\theta \rightarrow \pi - \theta$, corresponds to the exchange from a loop-loop correlator to a loop-antiloop correlator, where an antiloop is obtained from a given loop by exchanging the quark and the antiquark trajectories.)

3. Wilson-loop correlation functions on the lattice and comparison with known analytical results

The gauge-invariant Wilson-loop CF \mathcal{C}_E is a natural candidate for a lattice computation. In Refs. [23, 21] a Monte Carlo calculation of \mathcal{C}_E for several values of the relative angle and different configurations in the transverse plane has been performed, using 30000 quenched configurations generated with the SU(3) Wilson action at $\beta \equiv 6/g^2 = 6.0$, corresponding to a lattice spacing $a \simeq 0.1 \,\mathrm{fm}$, on a 16⁴ hypercubic lattice with periodic boundary conditions. The Wilson-loop CFs have been constructed using loops with transverse sizes $|\vec{r}_{1\perp}| = |\vec{r}_{2\perp}| = 1$ in lattice units $(\vec{R}_{i\perp} = a\vec{r}_{i\perp}, \vec{z}_{\perp} = a\vec{d}_{\perp})$ and seven different values of the relative angle θ , i.e., $\cot \theta = 0, \pm 1, \pm 2$. Without loss of generality (see the Appendix of Ref. [21]), the longitudinal-momentum fractions have been taken to be $f_1 = f_2 = \frac{1}{2}$: the loop configurations in the transverse plane that have been studied are $\vec{d}_{\perp} \parallel \vec{r}_{1\perp} \parallel \vec{r}_{2\perp}$ ("zzz") and $\vec{d}_{\perp} \perp \vec{r}_{1\perp} \parallel \vec{r}_{2\perp}$ ("zyy"). Also the orientation–averaged quantity ("ave") defined in Eq. (2.7) has been measured. Finally, the CFs have been calculated for the values $d \equiv |\vec{d}_{\perp}| = 0, 1, 2$ of the transverse distance between the centers of the loops: as expected (see the discussion in Section 4.1 below), the CFs vanish rapidly as d increases, thus making a "brute-force" Monte Carlo calculation very difficult at larger distances.

As already pointed out in the Introduction, numerical simulations of LGT can provide the Euclidean CF only for a finite set of θ -values, and so its analytic properties cannot be directly attained; nevertheless, they are first-principles calculations that give us (within the errors) the true QCD expectation for this quantity. Approximate analytical calculations of this same CF have then to be compared with the lattice data, in order to test the goodness of the approximations involved. This can be done either by direct comparison, when a numerical prediction is available, or by fitting the lattice data with the functional form provided by a given model. The Euclidean CFs we are interested in have been evaluated in the Stochastic Vacuum Model (SVM) [19], in perturbation theory (PT) [26, 17, 19], in the Instanton Liquid Model (ILM) [20, 21], and, using the AdS/CFT correspondence, for the $\mathcal{N}=4$ SYM theory at large N_c , large 't Hooft coupling and large distances between the loops [22], obtaining, respectively:

$$C_E^{(SVM)}(\theta) = \frac{2}{3} \exp\left[-\frac{1}{3}K_{SVM}\cot\theta\right] + \frac{1}{3}\exp\left[\frac{2}{3}K_{SVM}\cot\theta\right] - 1, \quad (3.1)$$

$$C_E^{(PT)}(\theta) = K_{PT} \cot^2 \theta, \tag{3.2}$$

$$C_E^{(\text{ILM})}(\theta) = \frac{K_{\text{ILM}}}{\sin \theta}, \tag{3.3}$$

$$C_E^{\text{(AdS/CFT)}}(\theta) = \exp\left[\frac{K_1}{\sin\theta} + K_2 \cot\theta + K_3 \cos\theta \cot\theta\right] - 1, \tag{3.4}$$

where the coefficients $K_i = K_i(\vec{z}_\perp; 1, 2)$ are functions of \vec{z}_\perp and of the dipole variables $\vec{R}_{i\perp}$, f_i . The comparison of the lattice data with these analytical calculations is not, generally speaking, fully satisfactory. The values of the chi–squared per degree of freedom $(\chi^2_{\text{d.o.f.}})$ of the best fits, performed in Ref. [23] using the above–reported functions (3.1)–(3.4), are listed in Table 1 (together with the values obtained from best fits with the parameterizations "Corr 1", "Corr 2" and "Corr 3", that we shall introduce and discuss in the next section). As one can see in Table 1, largely improved best fits have been obtained by combining the ILM and perturbative expressions into the following expression:

$$C_E^{(\text{ILMp})}(\theta) = \frac{K_{\text{ILMp1}}}{\sin \theta} + K_{\text{ILMp2}} \cot^2 \theta.$$
 (3.5)

As we have said in the Introduction, the main motivation in studying soft high–energy scattering is that it can lead to a resolution of the total cross section puzzle. From this point of view, the analytical models considered in this section are absolutely unsatisfactory, since they do not lead to rising, or, better, to *Froissart–like* total cross sections of

$\chi^2_{\rm d.o.f.}$	d = 0		d=1			d=2		
	zzz/zyy	ave	zzz	zyy	ave	zzz	zyy	ave
SVM	51	-	16	12	-	1.5	2.2	-
PT	53	34	16	13	13	1.5	2.2	4.5
ILM	114	94	14	15	45	0.45	0.35	1.45
ILMp	20	9.4	0.54	0.92	1.8	0.13	0.12	0.19
AdS/CFT	40	-	1	0.63	-	0.14	0.065	-
Corr 1	12.9	2.81	0.34	0.66	1.25	0.16	0.07	0.05
Corr 2	7.88	0.55	0.27	0.55	0.31	0.15	0.07	0.05
Corr 3	3.89	0.17	0.16	0.77	0.11	0.12	0.10	0.10

Table 1: Chi–squared per degree of freedom for a best fit with the indicated function: SVM [Eq. (3.1)], PT [Eq. (3.2)], ILM [Eq. (3.3)], ILMp [Eq. (3.5)], Corr 1 [Eq. (4.23)], Corr 2 [Eq. (4.26)], Corr 3 [Eq. (4.29)].

the form $B \log^2 s$ at high energy, as experimental data seem to suggest. In fact, the SVM, PT, ILM and ILMp parameterizations (3.1)–(3.3) and (3.5) lead to asymptotically constant total cross sections*, as it can be seen by using Eq. (2.6). Concerning the AdS/CFT expression (3.4), obtained in $\mathcal{N}=4$ SYM, it has been shown in [27] that, by combining the knowledge of the various coefficient functions K_i in (3.4) at large $|\vec{z}_{\perp}|$ [22] with the unitarity constraint in the small– $|\vec{z}_{\perp}|$ region, a non–trivial high–energy behavior for the dd total cross section can emerge (including a Pomeron–like behavior $\sigma \sim s^{1/3}$: note that, since a Conformal Field Theory has no $mass\ gap$, there is no need for the $Froissart\ bound$ to hold also in this case).

4. How a Froissart–like total cross section can be obtained: a new analysis of Wilson–loop correlators from lattice QCD

An ambitious question that one can ask at this point is if the lattice data are compatible with rising total cross sections. An answer can in principle be obtained by performing best fits to the lattice data with more general functions, leading to a non-trivial dependence on energy. This approach requires special care, because of the analytic continuation necessary to obtain the physical amplitude from the Euclidean CF: one has therefore to restrict the set of admissible fitting functions by imposing physical constraints, first of all *unitarity*.

Introducing the "hadron-hadron correlator" $\mathcal{C}_{M}^{(hh)}$ as the average of the "dipole-dipole correlator" \mathcal{C}_{M} over the dipole variables, weighted with the proper squared hadronic wave

^{*}Actually the ILM parameterization (3.3) leads to null total cross sections!

functions, i.e.,

$$\mathcal{C}_{M}^{(hh)}(\chi;|\vec{z}_{\perp}|) \equiv \int d^{2}\vec{R}_{1\perp} \int_{0}^{1} df_{1} |\psi_{1}(\vec{R}_{1\perp},f_{1})|^{2} \int d^{2}\vec{R}_{2\perp} \int_{0}^{1} df_{2} |\psi_{2}(\vec{R}_{2\perp},f_{2})|^{2} \times \mathcal{C}_{M}(\chi;\vec{z}_{\perp};\vec{R}_{1\perp},f_{1},\vec{R}_{2\perp},f_{2}), \tag{4.1}$$

one immediately recognizes from Eqs. (2.1) and (2.5) (see Refs. [13] and [15]) that $C_M^{(hh)}(\chi; |\vec{z}_{\perp}|)$ is nothing but the scattering amplitude $A(s, |\vec{z}_{\perp}|)$ in impact–parameter space (i.e., the partial–wave scattering amplitude $S_l - 1 = \eta_l e^{2i\delta_l} - 1$), which must satisfy the following well–known unitarity condition (see, for example, Refs. [28]): $|A(s, |\vec{z}_{\perp}|) + 1| \leq 1$. Therefore, this unitarity condition immediately translates to:

$$\left| \mathcal{C}_{M}^{(hh)}(\chi; |\vec{z}_{\perp}|) + 1 \right| \le 1.$$
 (4.2)

Since the hadronic wave functions are normalized to 1, the unitarity condition (4.2) is obviously satisfied if the following *sufficient* (and therefore "stronger") condition for the loop–loop correlator $C_M(\chi; \vec{z}_\perp; 1, 2)$ holds:

$$\left| \mathcal{C}_{M}(\chi; \vec{z}_{\perp}; \vec{R}_{1\perp}, f_{1}, \vec{R}_{2\perp}, f_{2}) + 1 \right| \le 1 \qquad \forall \ \vec{z}_{\perp}, \ \vec{R}_{1\perp}, \ \vec{R}_{2\perp}, \ f_{1}, \ f_{2},$$
 (4.3)

i.e., if the dipole–dipole correlator stays inside the Argand circle for all values of \vec{z}_{\perp} , $\vec{R}_{1\perp}$, $\vec{R}_{2\perp}$, f_1 , f_2 . It is also possible to find another ("weaker") sufficient unitarity condition, in terms of the averaged correlator C_M^{ave} , which is the Minkowskian version of C_E^{ave} , defined in Eq. (2.7) in the Section 2: in fact, as we have said there, we can substitute the correlator C_M^{ave} (without changing the result), whenever it is averaged over the dipole variables $\vec{R}_{i\perp}$ and f_i (with i=1,2) with the proper (squared) hadronic wave functions $|\psi_1|^2$ and $|\psi_2|^2$, as, for example, in Eq. (4.1). Therefore, the unitarity condition (4.2) is also satisfied if the following sufficient condition for the averaged loop–loop correlator (stronger than (4.2), but weaker than (4.3)) holds:

$$\left| \mathcal{C}_{M}^{ave}(\chi; |\vec{z}_{\perp}|; |\vec{R}_{1\perp}|, f_{1}, |\vec{R}_{2\perp}|, f_{2}) + 1 \right| \leq 1 \qquad \forall |\vec{z}_{\perp}|, |\vec{R}_{1\perp}|, |\vec{R}_{2\perp}|, f_{1}, f_{2}. \tag{4.4}$$

4.1. General considerations on the form of the correlator

In this section, we are going to introduce, and partially justify, new parameterizations of the CF that, in order: i) fit well the data; ii) satisfy the unitarity condition after analytic continuation; and iii) lead to total cross sections rising as $B \log^2 s$ in the high-energy limit (as experimental data seem to suggest).

We will show that the above conditions lead to rather strong constraints about the possible shape of the parameterization. First of all, we observe that the conditions ii) and iii) cannot be simultaneously satisfied by a correlator with the following factorized form: $-\text{Re }\mathcal{C}_M^{(hh)}(\chi;|\vec{z}_\perp|) = \varsigma(\chi)\upsilon(|\vec{z}_\perp|)$, with $\varsigma(\chi)$ rising with χ for $\chi \to \infty$, since the unitarity condition (4.2) implies that $0 \le -\text{Re}\mathcal{C}_M^{(hh)}(\chi;|\vec{z}_\perp|) \le 2$ (actually, with this factorized form, one can only have $\sigma_{\text{tot}}^{(hh)}(\chi) \to \text{constant for } \chi \to \infty$). This means that rising total cross sections can be obtained, without violating unitarity, only if the correlator is *not* factorizable. Let us now give a few general considerations about the form of the Euclidean loop–loop correlator. As a starting point, we shall assume that the Euclidean correlator can be written as:

$$C_E(\theta; \vec{z}_\perp; 1, 2) = \exp\left[K_E(\theta; \vec{z}_\perp; 1, 2)\right] - 1, \tag{4.5}$$

where $K_E(\theta; \vec{z}_\perp; 1, 2)$ is a real function (since the correlator C_E itself is known to be a real function [23]). This assumption, i.e., essentially the fact that: $C_E + 1 \ge 0$, is indeed rather well justified for many reasons: first, in the large- N_c expansion, the correlator C_E is expected to be of order $\mathcal{O}(1/N_c^2)$ (see Eq. (3.4) in Ref. [23]), so that $C_E + 1 \ge 0$ is certainly satisfied for large N_c ; moreover, all the known analytical models (SVM, ILM, AdS/CFT correspondence, perturbation theory,...) actually satisfy it; and last (but not least!), the lattice data obtained in Refs. [23, 21] confirm it.*

At this point, the Minkowskian correlator can be obtained after analytic continuation:

$$C_M(\chi; \vec{z}_\perp; 1, 2) = \exp\left[K_M(\chi; \vec{z}_\perp; 1, 2)\right] - 1,$$
 (4.6)

with $K_M(\chi; \vec{z}_\perp; 1, 2) = K_E(\theta \to -i\chi; \vec{z}_\perp; 1, 2)$. In the large- χ limit, the Minkowskian correlator C_M is expected to obey the unitarity condition (4.3), which using the parameterization (4.6), reduces to the following very simple relation:

$$\operatorname{Re} K_M(\chi; \vec{z}_\perp; 1, 2) \le 0 \quad \forall \vec{z}_\perp, \vec{R}_{i\perp}, f_i \quad (i = 1, 2). \tag{4.7}$$

So, the parameterizations that we are going to consider have the general form

$$C_E(\theta; \vec{z}_\perp; 1, 2) = \exp\left[\sum_i K_i(\vec{z}_\perp; 1, 2) F_{Ei}(\theta)\right] - 1,$$
 (4.8)

^{*}Actually, lattice data [23] seem to support an even stronger condition, i.e., the positivity of the correlator itself, $C_E \geq 0$, which, in terms of the parameterization (4.5) would mean $K_E \geq 0$. We are not aware of any rigorous proof of this condition, apart from a rough argument in terms of "minimal surfaces". (Let us observe also that this stronger condition is not expected to hold in the Abelian case: in fact, the parameterization for the Euclidean correlator C_E in quenched QED, found in Ref. [17], while still being of the form (4.5), has not the property $K_E \geq 0$.)

where the sum is over different terms with various functions $F_{Ei}(\theta)$ and various "parameters" $K_i(\vec{z}_{\perp}; 1, 2)$. We now make an important consideration about the dependence of the parameters K_i on the impact parameter $|\vec{z}_{\perp}|$. For a *confining* theory like QCD the loop–loop correlator C_E is expected to decay exponentially at large $|\vec{z}_{\perp}|$ as

$$C_E \sim \alpha e^{-\mu |\vec{z}_\perp|},$$
 (4.9)

where μ is some mass-scale proportional to the mass of the lightest glueball ($M_G \simeq 1.5$ GeV) or maybe (as suggested, for example, by the SVM model: see below) to the inverse $1/\lambda_{vac}$ of the so-called vacuum correlation length λ_{vac} , which has been measured with Monte Carlo simulations on the lattice in Refs. [29] (see also Ref. [30] for a review), both in the quenched ($\lambda_{vac} \simeq 0.22$ fm) and full QCD ($\lambda_{vac} \simeq 0.30$ fm). (For example, in the SVM model, see Eq. (3.1), one finds [19] that, for $|\vec{z}_{\perp}| \to \infty$, $K_{\text{SVM}} \sim e^{-|\vec{z}_{\perp}|/\lambda_{vac}}$, so that $C_E^{\rm SVM} \sim \frac{1}{9} K_{\rm SVM}^2 \cot^2 \theta \sim \alpha \, e^{-\mu |\vec{z}_\perp|}$, with $\mu = \frac{2}{\lambda_{vac}}$.) Therefore, we should require the same large- $|\vec{z}_{\perp}|$ behavior (4.9) for the parameters K_i in Eq. (4.8), i.e., $K_i \sim e^{-\mu|\vec{z}_{\perp}|}$. Instead, for a non-confining theory, e.g., for a conformal field theory, also different behaviors of the parameters K_i for large $|\vec{z}_{\perp}|$ are possible, typically like powers of $1/|\vec{z}_{\perp}|$. This is what happens for the parameters $K_{1,2,3}$ for the parameterization (3.4) obtained from the AdS/CFT correspondence [22, 27]. However, it can be shown that, in the general case, an asymptotic large- $|\vec{z}_{\perp}|$ behavior of the parameters K_i like powers of $1/|\vec{z}_{\perp}|$ leads to nonuniversal high-energy total cross sections, and can reproduce a Froissart-like behavior, $\sigma_{\rm tot}^{(hh)} \sim B \log^2 s$, only with very "ad hoc" dependencies of the parameters K_i on powers of $1/|\vec{z}_{\perp}|$ and of the functions $F_{Ei}(\theta)$ on powers of θ .

4.2. How a Froissart-like total cross section can be obtained

Let us now assume that the leading term for $\chi \to +\infty$ (i.e., for $s \to \infty$) of the Minkowskian dipole–dipole CF is of the form

$$C_M(\chi; \vec{z}_\perp; 1, 2) \underset{\chi \to +\infty}{\sim} \exp\left(i \beta f(\chi) e^{-\mu |\vec{z}_\perp|}\right) - 1, \tag{4.10}$$

where $\beta = \beta(1,2)$ is a function of the dipole variables and $f(\chi)$ is a positive and real function rising with χ , i.e, $f(\chi) \to +\infty$ for $\chi \to +\infty$: for example we can have $f(\chi) = e^{n\chi}$, $(\cosh \chi)^n$, $\chi^p e^{n\chi}$, It is then clear that, in order to satisfy the unitarity condition (4.3), the imaginary part of β has to be positive, i.e,

$$|\mathcal{C}_M(\chi; \vec{z}_\perp; 1, 2) + 1| \le 1 \quad \Leftrightarrow \quad \operatorname{Im} \beta \ge 0.$$
 (4.11)

The precise \vec{z}_{\perp} dependence of (4.10) is, of course, expected to be valid only for large enough $|\vec{z}_{\perp}|$. For simplicity, we shall first assume that it is valid $\forall |\vec{z}_{\perp}| \geq 0$. Following Eq. (2.6), and performing the change of variable $y = \mu |\vec{z}_{\perp}|$, we have that:

$$\sigma_{\text{tot}}^{(hh)} \sim \frac{4\pi}{\mu^2} \text{Re} \int d^2 \vec{R}_{1\perp} \int_0^1 df_1 |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 |\psi_2(\vec{R}_{2\perp}, f_2)|^2 I(\chi, \beta),$$
(4.12)

where the quantity $I(\chi, \beta)$ is defined as

$$I(\chi,\beta) \equiv \int_0^\infty dy \, y \left[1 - \exp\left(i \,\beta f(\chi) e^{-y}\right) \right]. \tag{4.13}$$

We can now expand the exponential in series, and exchange the order of integration and summation, obtaining:

$$I(\chi,\beta) = -\sum_{n=1}^{\infty} \frac{(i\beta f(\chi))^n}{n!} \int_0^{\infty} dy \, y \, e^{-ny} = -\sum_{n=1}^{\infty} \frac{(i\beta f(\chi))^n}{n! n^2}.$$
 (4.14)

The expression (4.14) is valid for an arbitrary function $f(\chi)$. However, as we have said, we are interested only in its asymptotic form for large χ , in the case of $f(\chi)$ rising with χ , i.e., $f(\chi) \to +\infty$ for $\chi \to +\infty$. Thus $\exists \chi_0 \in \mathbb{R}^+$ s.t. $f(\chi) > 0$, $\forall \chi \geq \chi_0$. So, for $\chi \geq \chi_0$, we can define the variable $\eta \equiv \log f(\chi)$, and re—write the quantity $I(\chi, \beta)$, Eqs. (4.13)–(4.14), as a function of η and β :

$$I(\chi,\beta) = J(\eta,\beta) \equiv -\sum_{n=1}^{\infty} \frac{(i\beta e^{\eta})^n}{n!n^2}.$$
 (4.15)

Now, deriving (4.15) with respect to the variable η , one finds

$$\frac{\partial J}{\partial \eta} = -\sum_{n=1}^{\infty} \frac{(i\beta e^{\eta})^n}{n!n}.$$
(4.16)

The sum of the above series is known (see, e.g, Ref. [31]) and given by

$$J'(\eta, \beta) = E_1(-i\beta e^{\eta}) + \log(-i\beta e^{\eta}) + \gamma, \quad -\pi < \arg(-i\beta e^{\eta}) < \pi, \tag{4.17}$$

where γ is the Euler–Mascheroni constant ($\gamma \simeq 0.57721...$) and $E_1(z)$ is the Schloemilch's exponential integral (see, e.g., Ref. [31]). Since $E_1(z) \sim e^{-z}/z$ at large |z|, for Re $z \geq 0$, and, moreover, Re $(-i\beta e^{\eta}) \geq 0 \Leftrightarrow \text{Im}\beta \geq 0$ is nothing but the unitarity condition (4.11), the asymptotic form of (4.17) is readily obtained:

$$J'(\eta, \beta) \underset{\eta \to +\infty}{\sim} \eta + \log(-i\beta) + \gamma + \mathcal{O}(e^{-\eta}). \tag{4.18}$$

Now we can re–integrate (4.18) in η , finding

$$J(\eta, \beta) \underset{\eta \to +\infty}{\sim} \frac{\eta^2}{2} + \eta \left(\log(-i\beta) + \gamma \right) + \text{constant} + \mathcal{O}(e^{-\eta}), \tag{4.19}$$

and, making the substitution $\eta = \log f(\chi)$,

$$I(\chi, \beta) \underset{\chi \to +\infty}{\sim} \frac{\log^2 f(\chi)}{2} + \log f(\chi) \left(\log(-i\beta) + \gamma \right) + \text{constant} + \mathcal{O}\left(\frac{1}{f(\chi)}\right).$$
 (4.20)

Let us observe the following important fact: the leading order in χ (and η) does not depend on β . So, coming back to Eq. (4.12), the asymptotic behavior of the total cross section turns out to be:

$$\sigma_{\text{tot}}^{(hh)} \sim \frac{4\pi}{\mu^2} \int d^2 \vec{R}_{1\perp} \int_0^1 df_1 \, |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 \, |\psi_2(\vec{R}_{2\perp}, f_2)|^2 \\ \times \left[\frac{\log^2 f(\chi)}{2} + \log f(\chi) \left(\log |\beta| + \gamma \right) + \dots \right], \tag{4.21}$$

where the whole dependence on the dipole variables $(\vec{R}_{i\perp}, f_i, i = 1, 2)$ is in the coefficient β . If one assumes a leading term of the type $f(\chi) = e^{n\chi}$ or, even more generally, $f(\chi) = \chi^p e^{n\chi}$ in the correlator, the resulting asymptotic behavior for the total cross section is (recalling that $\chi \simeq \log(s/m^2)$):

$$\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s, \quad \text{with:} \quad B = \frac{2\pi n^2}{\mu^2}.$$
 (4.22)

We want to emphasize the fact that the above result is universal, depending only on the mass scale μ , which, as we have said, sets the large- $|\vec{z}_{\perp}|$ dependence of the correlator. In fact, the integration over the dipole variables does not affect the coefficient of the leading term, since the hadronic wave functions are normalized to 1. Let us also observe, that the universal coefficient B is not affected by the masses of the scattering particles. In fact, in the case of scattering of two different mesons with masses m_1 and m_2 , the rapidity is $\chi \sim \log(\frac{s}{m_1 m_2})$. Therefore, considering the case of two different mesons (or, in general, a change of the energy scale implicitly contained in Eq. (4.22)) does not affect the universal coefficient B of the leading term (but will in general affect sub-leading log and constant terms), since $\chi \sim \log(\frac{s}{m_1 m_2}) = \log(\frac{s}{s_0}) + \log(\frac{s_0}{m_1 m_2})$, with $\sqrt{s_0}$ being an arbitrary energy scale. This is the main theoretical achievement in this work.

This same relation can be derived also with less stringent conditions on the \vec{z}_{\perp} dependence, assuming (as it must be!) the exponential-type dependence in Eqs. (4.9) and

(4.10) only for $|\vec{z}_{\perp}| > z_0$, with z_0 much larger than $1/\mu$ and the dipole sizes. Starting from the expression (2.6) of the total cross section, re–written as $\sigma_{\text{tot}}^{(hh)} = -4\pi \text{Re} \int_0^{\infty} |\vec{z}_{\perp}| \, d|\vec{z}_{\perp}| \, \mathcal{C}_M^{(hh)}(\chi; |\vec{z}_{\perp}|)$, where $\mathcal{C}_M^{(hh)}(\chi; |\vec{z}_{\perp}|)$ has been defined in Eq. (4.1), one can split the integration in the variable $|\vec{z}_{\perp}|$ in two parts, a "tail" contribution $(\int_{z_0}^{\infty} d|\vec{z}_{\perp}|\dots)$, which can be evaluated using the approximate expression (4.10) for the loop–loop correlator, and a "core" contribution $(\int_0^{z_0} d|\vec{z}_{\perp}|\dots)$, which, instead, can be bounded using the unitarity condition (4.2) [or (4.3)–(4.4)], in the form: $-2 \leq \text{Re} \, \mathcal{C}_M \leq 0$.

In the next subsection we are going to show our new analysis of the lattice data.

4.3. New parameterizations for the correlator

In what follows we show some parameterizations that we have found that satisfy the criteria i)-iii) listed above, together with the corresponding estimate of the asymptotic term of the high-energy total cross sections. Despite the appearances, it has not been simple to find such parameterizations: what follows is a selection between more than 70 different parameterizations that we have tried. For each proposed parameterization of the correlator, we are going to show the $\chi^2_{d.o.f.}$ of the corresponding best fit to the lattice data for each given transverse distance and each given configuration: the results are summarized in Table 1. As we have already pointed out in Section 2, the averaged correlator \mathcal{C}^{ave} is somehow "closer" to the hadron-hadron scattering matrix $\mathcal{M}_{(hh)}$ than the correlator at fixed transverse configuration (like "zzz" or "zyy"), since it is actually the result of an integration of the dipole-dipole correlator over the orientations of the dipoles. The analysis performed above in Sections 4.1 and 4.2 can be repeated for the averaged correlator \mathcal{C}^{ave} without altering any conclusions. For this reason we are going to focus our analysis on the averaged correlator \mathcal{C}^{ave} only; however, the $\chi^2_{d.o.f.}$ of the best fits to the "zzz" and the "zyy" data are also shown for comparison in Table 1. Since, as we have already said in Section 2, the averaged correlator C^{ave} is automatically crossingsymmetric, so are the parameterizations that we propose.

[†]However, it was observed in Ref. [23] that a small (but nonzero!) Odderon (C-odd) contribution in dd scattering, which is related through the crossing-symmetry relations [25] to the antisymmetric part of $C_E(\theta)$ with respect to $\theta = \frac{\pi}{2}$, is present in the lattice data corresponding to the "zzz" and "zyy" transverse configurations. As noticed there, a crossing-antisymmetric term in the exponent K_E of the Euclidean CF (4.5) proportional to $\cot \theta$ (as, for example, the one appearing in the SVM parameterization (3.1) and also the one appearing in the AdS/CFT parameterization (3.4)) is in general suitable for taking into account this Odderon contribution and fits quite well the lattice data for the antisymmetric part of the correlator. Let us note also that its analytic continuation, $\cot \theta \to i \coth \chi$, is limited for $\chi \to \infty$, and

With regard to the explicit angular dependence of the possible terms $F_{Ei}(\theta)$, let us make some preliminary considerations. It is known [23, 21] that lattice data for the correlator blow up at $\theta = 0^{\circ}$, 180° (as expected from the relation between the correlator and the dipole–dipole static potential [32]), and that they are clearly different from zero for $\theta = 90^{\circ}$. A simple term like $1/\sin\theta$ (which always comes out as a Jacobian in the integration over the longitudinal coordinates in the analytical models considered in the previous section) can account for such a behavior and, as noticed in Refs. [23, 21], it fits quite well the data around $\theta = 90^{\circ}$. Therefore, we shall always include a $1/\sin\theta$ term in our parameterizations for the exponent in Eq. (4.8).

Concerning the analysis of the impact–parameter dependence, we want to stress the fact that it must be taken only as an estimate, since only a few (small) values of the impact parameter are available from the lattice data ($|\vec{z}_{\perp}| = ad$, with d = 0, 1, 2).

4.3.1. Correlator 1

Following a first possible strategy, we have tried to improve best fits achieved with the ILMp expression (3.5): the idea is to combine known QCD results and variations thereof. As an example, one could consider exponentiating the two–gluon exchange and the one–instanton contribution (i.e., the ILMp expression), and supplementing it with a term which could yield a rising cross section, e.g., a term proportional to $\cos\theta \cot\theta$, like the one present in the AdS/CFT parameterization (3.4). We thus find the following parameterization:

$$C_E(\theta) = \exp\left[\frac{K_1}{\sin\theta} + K_2 \cot^2\theta + K_3 \cos\theta \cot\theta\right] - 1, \tag{4.23}$$

whose Minkowskian counterpart is:

$$C_M(\chi) = \exp\left[i\left(\frac{K_1}{\sinh\chi} + K_3\cosh\chi\coth\chi\right) - K_2\coth^2\chi\right] - 1. \tag{4.24}$$

The unitarity condition (4.7) is satisfied if $K_2 \geq 0$: from Table 2 one sees that the parameter K_2 obtained from a best fit satisfies this condition, within the errors. The best–fit functions are plotted in Fig. 2. Performing a best fit with an exponential function $\sim e^{-\mu|\vec{z}_{\perp}|}$ over the three distances, one finds that the coefficient K_3 of the leading term for $\chi \to \infty$ has a mass–scale $\mu = 4.64(2.38)$ GeV, that, following the result (4.22) of the

so it is consistent with the *Pomeranchuk theorem*, at least for rising total cross sections.

Corr 1	d = 0	d = 1	d=2
K_1	$5.85(42) \cdot 10^{-3}$	$3.07(37) \cdot 10^{-3}$	$8.7(3.1) \cdot 10^{-4}$
K_2	$9.60(98) \cdot 10^{-2}$	$2.44(49) \cdot 10^{-2}$	$-5.3(84.5) \cdot 10^{-5}$
K_3	$-7.8(1.3) \cdot 10^{-2}$	$-1.37(72) \cdot 10^{-2}$	$1.7(1.9) \cdot 10^{-3}$
$\chi^2_{\rm d.o.f.}$	2.81	1.25	0.05
Corr 2	d = 0	d = 1	d=2
K_1	$6.03(42) \cdot 10^{-3}$	$3.26(38) \cdot 10^{-3}$	$8.7(3.2) \cdot 10^{-4}$
K_2	$4.63(46) \cdot 10^{-1}$	$1.33(25) \cdot 10^{-1}$	$-1.2(54.2)\cdot 10^{-4}$
K_3	$-4.54(50) \cdot 10^{-1}$	$-1.26(28) \cdot 10^{-1}$	$1.7(6.7) \cdot 10^{-3}$
$\chi^2_{\rm d.o.f.}$	0.55	0.31	0.05
Corr 3	d = 0	d = 1	d=2
K_1	$6.02(36) \cdot 10^{-3}$	$3.46(29) \cdot 10^{-3}$	$1.07(20) \cdot 10^{-3}$
K_2	$1.29(5) \cdot 10^{-1}$	$4.47(27) \cdot 10^{-2}$	$2.11(73) \cdot 10^{-3}$
$\chi^2_{\rm d.o.f.}$	0.17	0.11	0.10

Table 2: Parameters (with their errors) for the Correlators 1 [Eq. (4.23)], 2 [Eq. (4.26)], and 3 [Eq. (4.29)], obtained from best fits to the averaged lattice data, and the corresponding $\chi^2_{\text{d.o.f.}}$, for the transverse distances d = 0, 1, 2.

previous section (with n = 1, as implied by this parameterization), leads to the following asymptotic total cross section:

$$\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$$
, with: $B = 0.113_{-0.037}^{+0.364} \text{ mb.}$ (4.25)

This is compatible, within the large errors, with the experimental result $B_{exp} \simeq 0.3$ mb reported in the Introduction.

4.3.2. Correlator 2

Another possible strategy is suggested again by the AdS/CFT expression (3.4): one can try to adapt to the case of QCD the analytical expressions obtained in related models, such as $\mathcal{N}=4$ SYM. Although, of course, Eq. (3.4) is not expected to describe QCD, it is sensible to assume in this case a similar functional form (basically assuming the existence of the yet unknown gravity dual for QCD). Assuming moreover that the known power–law behavior of the K_i 's (expected for a conformal theory) goes over into an exponentially damped one (expected for a confining theory), $K_i \sim e^{-\mu |\vec{z}_\perp|}$, one obtains a Froissart–like total cross section $\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$. In this spirit, the second parameterization that we

propose is:

$$C_E(\theta) = \exp\left[\frac{K_1}{\sin\theta} + K_2\left(\frac{\pi}{2} - \theta\right)\cot\theta + K_3\cos\theta\cot\theta\right] - 1. \tag{4.26}$$

This one contains, in addition to the usual AdS/CFT-like terms $1/\sin\theta$, $\cot\theta$ and $\cos\theta\cot\theta$, also another term proportional to $\theta\cot\theta$: the coefficients of the terms $\cot\theta$ and $\theta\cot\theta$ are constrained by requiring that $C_E(\theta)$ is crossing symmetric. The analytic continuation of (4.26) is

$$C_M(\chi) = \exp\left[i\left(\frac{K_1}{\sinh\chi} + K_2\frac{\pi}{2}\coth\chi + K_3\cosh\chi\coth\chi\right) - \chi K_2\coth\chi\right] - 1. \quad (4.27)$$

The unitarity condition (4.7) becomes, in this case, $K_2 \geq 0$, which is satisfied by the best-fit parameter within the errors (see Table 2). The best-fit functions are plotted in Fig. 3. After the best fit over the distances with an exponential function, one finds for the leading-term coefficient K_3 a mass-scale $\mu = 3.79(1.46)$ GeV. Thus, by virtue of Eq. (4.22) (with n = 1, as implied by this parameterization), this correlator leads to the following asymptotic total cross section:

$$\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$$
, with: $B = 0.170_{-0.081}^{+0.277} \text{ mb}$, (4.28)

that is again compatible, within the large errors, with the experimental result.

4.3.3. Correlator 3

The last parameterization that we are going to propose is:

$$C_E(\theta) = \exp\left[\frac{K_1}{\sin\theta} + K_2\left(\frac{\pi}{2} - \theta\right)^3 \cos\theta\right] - 1. \tag{4.29}$$

The first term is the usual $1/\sin\theta$, while the second one is less "familiar", in the sense that it is not present in the analytical models known in the literature: but is a fact that, using this parameterization, the best fit is extremely good (see Table 1), even if it has only two parameters. The Minkowskian version of the correlator (4.29) is

$$C_M(\chi) = \exp\left[i\left(\frac{K_1}{\sinh \chi} + K_2 \cosh \chi \left(\frac{3}{4}\pi^2 \chi - \chi^3\right)\right) + K_2 \cosh \chi \left(\frac{\pi^3}{8} - \frac{3}{2}\pi \chi^2\right)\right] - 1. \quad (4.30)$$

[‡]Although such a term could seem "strange" at first sight, not being a "simple" combination of $\sin \theta$ and $\cos \theta$, it has been shown in the first Ref. [16], through an explicit calculation up to the order $\mathcal{O}(g^4)$ in perturbation theory, that a similar term actually shows up in the case of the CF of two Wilson lines.

The unitarity condition (4.7) reduces (in the large- χ limit) to $K_2 \geq 0$, that is fully satisfied by the best-fit parameter shown in Table 2. The best-fit functions are plotted in Fig. 4. As regards the total cross section, let us note that in this case the leading term (for $\chi \to +\infty$) in the exponent in (4.30) is of the form $\chi^3 e^{\chi}$: so, we can find the asymptotic behavior of the total cross section simply taking $f(\chi) = \chi^3 e^{\chi}$ (i.e., with our notation, n = 1 and p = 3) in the expression (4.21), that leads again to the leading behavior reported in Eq. (4.22) (with n = 1).

After an exponential best fit over the distances, one finds that the mass–scale of the leading–term coefficient K_2 is $\mu = 3.18(98)$ GeV. Thus, the asymptotic total cross section, derived from this correlator, reads:

$$\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$$
, with: $B = 0.245^{+0.263}_{-0.100} \text{ mb}$. (4.31)

The comparison with the experimental asymptotic coefficient is extremely good and seems better than the previous ones, even if the errors are always very large.

5. Conclusions

The nonperturbative approach to *soft* high–energy hadron–hadron (dipole–dipole) scattering, based on the analytic continuation of Wilson–loop CFs from Euclidean to Minkowskian theory, makes possible the investigation of the problem of the asymptotic energy dependence of hadron–hadron total cross sections from the point of view of lattice QCD, by means of Monte Carlo numerical simulations.

In this paper we have performed a new analysis of the data for the Wilson-loop correlator, originally obtained in Refs. [23, 21] by Monte Carlo simulations in Lattice Gauge Theory, and, in particular, Section 4 has been focused on the search for a new parameterization of the (Euclidean) correlator that, in order: i) fits well the lattice data; ii) satisfies (after analytic continuation) the unitarity condition; and, most importantly, iii) leads to a rising behavior of total cross sections at high energy, in agreement with experimental data. In particular, one is interested in the dependence of the correlation function on the angle θ between the loops, since it is related, after analytic continuation, to the energy dependence of the scattering amplitudes, and also in its dependence on the impact–parameter distance. In Section 4 we have shown that, making some reasonable assumptions about the angular dependence and the impact–parameter dependence of the various terms in the parameterization, our approach leads quite "naturally" to total cross

	$\mu \; (\text{GeV})$	$\lambda = \frac{1}{\mu} \text{ (fm)}$	$B = \frac{2\pi}{\mu^2} \text{ (mb)}$
Corr 1	4.64(2.38)	$0.042^{+0.045}_{-0.014}$	$0.113^{+0.364}_{-0.037}$
Corr 2	3.79(1.46)	$0.052^{+0.032}_{-0.014}$	$0.170^{+0.277}_{-0.081}$
Corr 3	3.18(98)	$0.062^{+0.028}_{-0.015}$	$0.245^{+0.263}_{-0.100}$

Table 3: Comparison of the mass–scale μ , the "decay length" $\lambda = 1/\mu$ and the coefficient $B = 2\pi/\mu^2$ derived from our parameterizations.

sections rising asymptotically as $B \log^2 s$ (that is what experimental data seem to suggest). Moreover, in our approach the coefficient B turns out to be universal, i.e, the same for all hadronic scattering processes (as it also seems to be suggested by experimental data), being related (see Eq. (4.22)) to the mass–scale μ which sets the large impact–parameter exponential behavior of the correlator: this type of behavior is typical of a confining theory, like QCD, and μ is expected to be proportional to the lightest glueball mass M_G or to the inverse of the so–called " $vacuum\ correlation\ length$ " λ_{vac} . This is actually the main result of this paper.

Concerning the comparison between the numerical data obtained from the best fits and the experimental value of B, the agreement is quite good since the values are compatible within the large errors. In Table 3 we report the mass-scale μ (and the decay length $\lambda =$ $1/\mu$) derived from the parameterizations that we have considered in the previous section, together with the predicted universal coefficient $B=2\pi/\mu^2$. However, we want to remark the fact that the values that we have found have been obtained from a limited set of "short" (i.e., surely not asymptotic!) distances, and so they must be taken only as an estimate. Of course, when more lattice data (at larger distances) will be available, the relation $B = 2\pi/\mu^2$ (derived from Eq. (4.22) with n = 1, as occurs in our parameterizations of the correlator) may be confirmed or not. Concerning the relation between μ and the inverse of the vacuum correlation length λ_{vac} or the lightest glueball mass M_G , at present a rigorous analytical determination in QCD is lacking (apart from the result $\mu = 2/\lambda_{vac}$ obtained in the SVM model) and would be surely an important and helpful result. Using for μ an estimate derived from the experimental value of B, i.e, $\mu_{exp} = \sqrt{2\pi/B_{exp}} \simeq 2.85 \text{ GeV}$ (as we have said above, our numerical estimates for μ are compatible with μ_{exp} within the large errors, as shown in Table 3), one finds that $\mu_{exp} \sim (3 \div 4)/\lambda_{vac}$ or $\mu_{exp} \sim 2M_G$. Of course, only further investigations (both numerical and analytical) can confirm (or not) these results. In this respect, we must also remark that the whole analytic derivation of the result in Eq. (4.22) is, of course, intended to be performed in full QCD (including

dynamical quarks), while the subsequent numerical analysis has been performed using the lattice data for the Wilson-loop correlation function which are available at the moment, and which were obtained in *quenched* (i.e., *pure-qauge*) QCD. However, we expect that the mass-scale μ , which enters Eq. (4.22), is essentially gluonic, being related in some way (as we have said) to the vacuum correlation length λ_{vac} or to the lightest glueball mass M_G , and so it should not dramatically change when including dynamical quark effects (hopefully, in near-future full-QCD lattice computations of the Wilson-loop correlator). Pushing this "speculation" a little bit further, we indeed expect, just on the basis of the experience with the vacuum correlation length λ_{vac} (which increases from 0.22 fm in quenched QCD up to about 0.30 fm in full QCD), that the inclusion of dynamical quark effects should improve the agreement between the theoretical determination of μ (i.e., of the decay length $\lambda = 1/\mu$, and of the parameter $B = 2\pi/\mu^2$: see the results in Table 3) and its experimental value $\mu_{exp} \simeq 2.85$ GeV (corresponding to $\lambda_{exp} \simeq 0.07$ fm and $B_{exp} \simeq 0.3 \text{ mb}$), since λ is expected to increase a little bit, so that μ should decrease and the parameter B should increase, moving towards the experimental value. Of course, it would be desirable to have lattice results in full QCD ...

Finally, let us observe that the functional integral approach turns out to be fundamental for achieving this result: in fact, the investigation of hadron-hadron elastic scattering in the soft regime is mainly founded on the elementary loop-loop CF, which is then folded with some proper wave functions for the specific hadrons involved in the scattering process. Therefore, it is "natural" to expect that a universal behavior of the hadronic total cross sections at high energy may be originated by the loop-loop correlator itself: and actually it has been so. Of course, strictly speaking, our approach, based on the looploop CF, and the corresponding conclusion about the universality of B, only applies to meson-meson scattering: in this sense, we can consider as a real prediction the fact that the value of B that we have found from our analysis is consistent (within the errors) with the experimental value B_{exp} , which has been found considering baryon-baryon (mainly, pp and $p\bar{p}$) and meson-baryon scattering. However, as briefly recalled at the beginning of Section 2, also for the treatment of baryons a similar, but more involved, picture can be adopted, using a genuine three-body configuration or, alternatively and even more simply, a quark-diquark configuration [11, 12, 13, 14, 15]. In particular, adopting a quark-diquark configuration for baryons, we can directly extend our approach (based on the loop-loop correlator) and the corresponding results to include also the case of baryon-baryon and meson-baryon scattering. This is probably enough to yield the Pomeron (C-even) contribution (going as $B \log^2 s$) to hadron-hadron scattering, but not enough to consider also possible Odderon (C-odd) contributions, which are sub-leading in the high-energy limit, due to the Pomeranchuk theorem. In fact, as already noticed above (in Section 4.3), these C-odd contributions are averaged to zero in dipole-dipole scattering, since in this case the relevant CF C_E^{ave} is automatically crossing-symmetric, and so they are probably visible only adopting a genuine three-body configuration for baryons.

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Figure captions

- Fig. 1 The space—time configuration of the two Wilson loops W_1 and W_2 entering the expression for the dipole–dipole elastic scattering amplitude in the high–energy limit.
- Fig. 2 Comparison of lattice data for the averaged correlator to best fits with the parameterization (4.23) (Correlator 1).
- **Fig. 3** Comparison of lattice data for the averaged correlator to best fits with the parameterization (4.26) (Correlator 2).
- **Fig. 4** Comparison of lattice data for the averaged correlator to best fits with the parameterization (4.29) (Correlator 3).

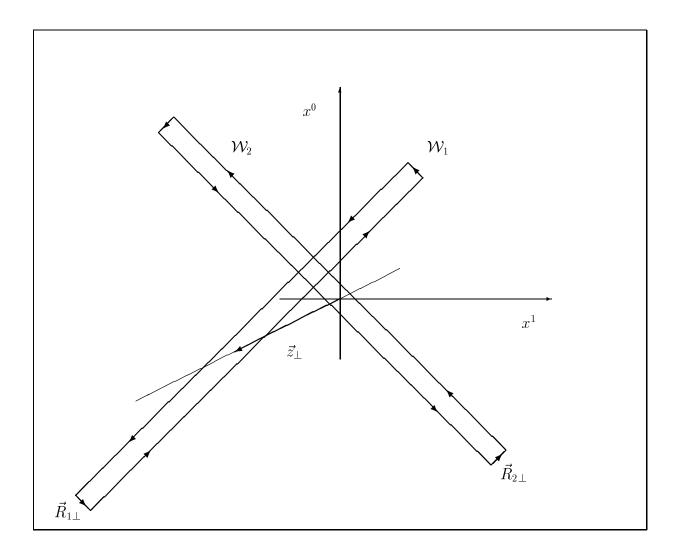


Figure 1: The space–time configuration of the two Wilson loops W_1 and W_2 entering the expression for the dipole–dipole elastic scattering amplitude in the high–energy limit.

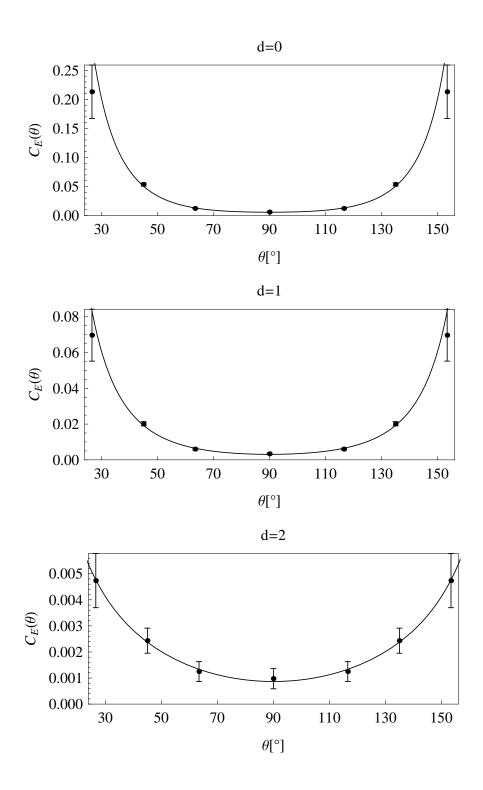


Figure 2: Comparison of lattice data for the averaged correlator to best fits with the parameterization (4.23) (Correlator 1).

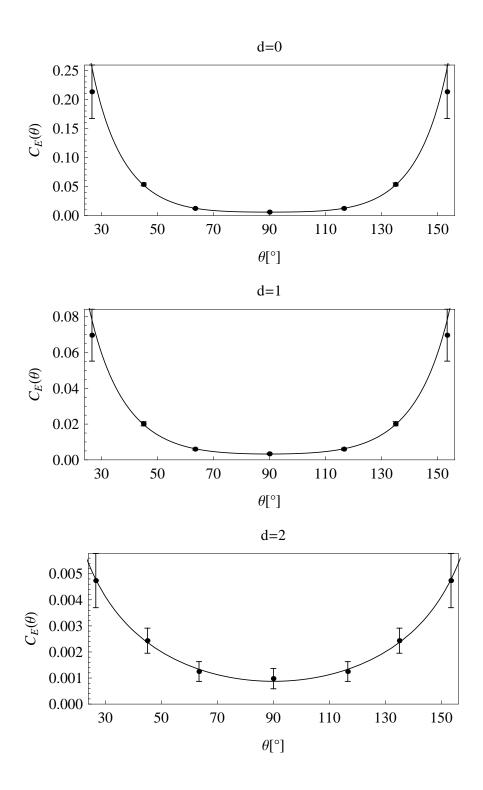


Figure 3: Comparison of lattice data for the averaged correlator to best fits with the parameterization (4.26) (Correlator 2).

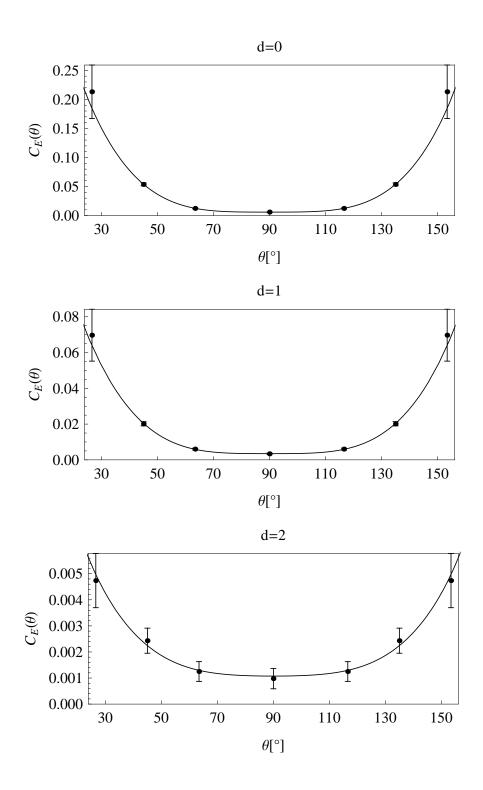


Figure 4: Comparison of lattice data for the averaged correlator to best fits with the parameterization (4.29) (Correlator 3).